HOMEWORK 6

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ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P192. # 23.1

Proof. **(b).** $a_n = \frac{1}{n^n}$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \frac{1}{n} = 0.$$

So $\beta = 0$. $R = \frac{1}{\beta} = \infty$. Thus this series converges for all x.

(d). $a_n = \frac{n^3}{3^n}$. Then

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \limsup_{n\to\infty} \frac{(n+1)^3}{n^3} \times \frac{3^n}{3^{n+1}} = \frac{1}{3}.$$

Thus

$$\beta = \frac{1}{3}, R = \frac{1}{\beta} = 3.$$

Then the radius of convergence is 3.

For $x = \pm 3$, $\lim_{n\to\infty} (\pm 1)^n n^3$ either goes to ∞ or does not exist. So the power series diverges at ± 3.

Thus the exact interval of convergence is $(-3, 3)$.

f. $a_n = \frac{1}{(n+1)^2 2^n}$.

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2 2^n}{(n+2)^2 2^{n+1}} = \frac{1}{2}.$$

Thus $\beta = \frac{1}{2}$ and $R = 2$. So the radius of convergence is 2. For $x = 2$, the power series reduces to $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$, which converges.

For $x = -2$,it is $\sum \frac{(-1)^n}{(n+1)^2}$, which also converges. Thus the exact interval of convergence is $[-2, 2]$.

(h). $a_n = \frac{(-1)^n}{n^2 4^n}$.

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \limsup_{n\to\infty} \frac{n^2 4^n}{(n+1)^2 4^{n+1}} = \frac{1}{4}.$$

So $\beta = \frac{1}{4}$. Then $R = 4$, which is the radius of convergence. For the exact interval of convergence, we consider ± 4. For $x = 4$, the power series reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges. For $x = -4$, the power series reduces to

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges. Therefore the exact interval of convergence is $[-4, 4]$. □

2. P192. #23.2

Proof. **(a).** $a_n = \sqrt{n}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1.$$

Thus $\beta = 1$, $R = \frac{1}{\beta} = 1$. This implies that the radius of convergence is 1. For $x = \pm 1$, the limit of the sequence does not converge to zero. So the exact interval of convergence is $(-1, 1)$.

(b). $a_n = \frac{1}{n\sqrt{n}}$.

(c).

$$a_k = \begin{cases} n!, & \text{for } k = n!, \\ 0, & \text{for other } k. \end{cases}$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1.$$

So $\beta = 1$. Then $R = 1$, which is the radius of convergence. For $x = \pm 1$, the limit of the sequence does not converge to zero. Hence the exact interval of convergence is $(-1, 1)$.

(d).

$$a_n = \begin{cases} \frac{3^n}{\sqrt{n}}, & \text{for } k = 2n+1, \\ 0, & \text{for other } n. \end{cases}$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[2n+1]{\frac{3^n}{\sqrt{n}}} = \lim_{n\to\infty} \frac{3^{\frac{n}{2n+1}}}{n^{\frac{1}{2(2n+1)}}} = \sqrt{3}.$$

So $\beta = \sqrt{3}$. Then $R = \frac{1}{\sqrt{3}}$, which is the radius of convergence. For $x = \frac{1}{\sqrt{3}}$, the power series reduces to

$$\sum \frac{1}{\sqrt{3n}}$$

. Hence the power series diverges. For $x = -\frac{1}{\sqrt{3}}$, the power series reduces to

$$-\sum \frac{1}{\sqrt{3n}}.$$

Hence the power series diverges, either. So the exact interval of convergence is $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. □

3. P192. #23.4

Proof. **(a).**For $a_n = \left(\frac{4+2(-1)^n}{5}\right)^n$,

$$\limsup_{n\to\infty} (a_n)^{1/n} = \limsup_{n\to\infty} \frac{4+2(-1)^n}{5} = \frac{6}{5}.$$

$$\liminf_{n\to\infty} (a_n)^{1/n} = \limsup_{n\to\infty} \frac{4+2(-1)^n}{5} = \frac{2}{5}.$$

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left(\frac{4+2(-1)^{n+1}}{5}\right)^{n+1} \left(\frac{5}{4+2(-1)^n}\right)^n = \infty.$$

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left(\frac{4+2(-1)^{n+1}}{5}\right)^{n+1} \left(\frac{5}{4+2(-1)^n}\right)^n = 0.$$

(b). Both series diverge because

$$\limsup_{n\to\infty} (a_n)^{1/n} = \limsup_{n\to\infty} \frac{4+2(-1)^n}{5} = \frac{6}{5}.$$

(c). For the power series, $\sum a_n x^n$, the radius of convergence is $\frac{5}{6}$. For $x = \frac{5}{6}$, the power series is

$$\sum \left(\frac{4+2(-1)^n}{6}\right)^n.$$

It diverges because the limit of the sequence does not converge to zero,

$$\limsup_{n\to\infty} \left(\frac{4+2(-1)^n}{6}\right)^n = 1.$$

Likewise for $x = -\frac{5}{6}$. So the exact interval of convergence is $(-\frac{5}{6}, \frac{5}{6})$. □

4. P192.#23.5

Proof. **(a).** We prove it by contradiction. If $R > 1$, then for $|x| < R$, the series converges for $|x| < R$. We take x_0 such that $1 < x_0 < R$. Then for the series $\sum a_n x_0^n$, if all a_n are integers and if infinitely many of them are nonzero, then the limit of the sequence may not exist; or if it exists, it is not zero. Indeed, the series $\sum a_n x_0^n$ converges. Then

$$lim_{n\to\infty} a_n x_0^n = 0.$$

This implies

$$\lim_{n\to\infty} |a_n x_0^n| = 0.$$

A contradiction. So $R \leq 1$.

(b). If $\limsup |a_n| > 0$, then there exists a subsequence a_{n_k} such that

$$\lim_{k\to\infty} |a_{n_k}| = \limsup_{n\to\infty} |a_n|.$$

So $\lim_{k\to\infty} |a_{n_k}| > 0$. Next the proof goes similarly as in part (a). We prove it by contradiction. If $R > 1$, then for $|x| < R$, the series converges for $|x| < R$. We take x_0 such that $1 < x_0 < R$. Then the series $\sum a_n x_0^n$ converges. Then

$$lim_{n\to\infty} a_n x_0^n = 0.$$

This implies

$$\lim_{n\to\infty} |a_n x_0^n| = 0.$$

In particular,

$$\lim_{k\to\infty} |a_{n_k} x_0^{n_k}| = 0.$$

A contradiction. So $R \leq 1$. □

5. P193. # 23.6

Proof. **(a).** This follows from comparison test. We know that $\sum a_n R^n$ converges, then

$$|a_n(-R)^n| = a_n R^n.$$

So $\sum a_n(-R)^n$ converges.

(b). $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$. The exact interval of convergence for this power series is $(-1, 1]$. □

6. P230. # 28.1

Proof. Since the absolute value function $f(x) = |x|$ is not differentiable at zero. So for (a), the set of the points where it is not differential is $\{0\}$.

For (b), the set of points where it is not differentiable is x such that

$$\sin x = 0,$$

i.e.,

$$x \in \{k\pi : k \in \mathbb{Z}\}.$$

For (c), the set of points where it is not differentiable is x such that

$$x^2 - 1 = 0,$$

i.e.,

$$x \in \{1, -1\}.$$

□

7. P30. #28.2

Proof. **(a).** We compute

$$\lim_{h \to 0} \frac{(2+h)^3 - 2^3}{h} = \lim_{h \to 0} \frac{6h(h+2) + h^3}{h} = \lim_{h \to 0} 6(h+2) + h^2 = 12.$$

(b).

$$\lim_{h \to 0} \frac{(a+h+2) - (a+2)}{h} = 1.$$

(c).

$$\lim_{h \to 0} \frac{(h+0)^2 \cos(h+0) - 0^2 \cos 0}{h} = \lim_{h \to 0} h \cos h = 0.$$

(d).

$$\lim_{h \to 0} \frac{\frac{3(1+h)+4}{2(1+h)-1} - 7}{h} = -\lim_{h \to 0} \frac{11}{2h+1} = -11.$$

□

8. P230. # 28.3

Proof. **(a).** For $a > 0$,

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}},$$

where $\sqrt{x}-\sqrt{a} = \frac{x-a}{\sqrt{x}+\sqrt{a}}$.

(b). For $a \neq 0$,

$$\lim_{x\to a} \frac{x^{1/3}-a^{1/3}}{x-a} = \lim_{x\to a} \frac{1}{x^{3/2}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{1}{3a^{2/3}}$$

because $x - a = (x^{1/3} - a^{1/3})(x^{3/2} + x^{1/3}a^{1/3} + a^{2/3})$.

(c). No. We consider the definition. We write

$$\lim_{h\to 0} \frac{(0+h)^{1/3}-0}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty,$$

which is not finite. So the derivative of $x^{1/3}$ at 0 does not exist. □

9. P230. # 28.4

Proof. **(a).**We know that $\sin\frac{1}{x}$ is a composition of two functions, $\sin x$ and $\frac{1}{x}$, the latter of which is differentiable at $x \neq 0$. $\sin x$ is differentiable everywhere. So $\sin\frac{1}{x}$ is differentiable at $x \neq 0$ by Theorem 28.4. By Theorem 28.3, $x^2 \sin\frac{1}{x}$ is differentiable at $x \neq 0$.

$$f'(a) = 2a \sin\frac{1}{a} - \cos\frac{1}{a}.$$

(b). We compute

$$\lim_{h\to 0} \frac{(0+h)^2 \sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h \sin\frac{1}{h} = 0.$$

(c). Since $\cos 1a$ is not continuous at $a = 0$, $f'(a)$ is not continuous at $a = 0$.

□

10. P231. # 28.7

Proof. Part (a) is skipped. For part (b), since $f(0) = 0$,

$$\lim_{h \to 0} \frac{(0+h)^2 - 0}{h} = \lim_{h \to 0} h = 0.$$

(c). For $x > 0$, $f'(x) = 2x$. For $x < 0$, $f'(x) = 0$.

(d). f' is clearly continuous at $x = 0$. So f' is continuous on $\mathbb{R}$. However, it is not differentiable at $x = 0$ because

$$\lim_{h \to 0+} \frac{f'(0+h) - f'(0)}{h} = 2,$$

and

$$\lim_{h \to 0-} \frac{f'(0+h) - f'(0)}{h} = 0.$$

□

11. P231. # 28.11

Proof. By the chain rule twice,

$$(h \circ g \circ f)'(a) = h'(g \circ f(a)) \times g'(f(a))f'(a).$$

□

12. P231. # 28.12

Proof. We know that

$$\cos' x = -\sin x, (e^x)' = e^x, \text{ and } (x^n)' = nx^{n-1},$$

By chain rule,

$$\frac{d\cos(e^{x^5-3x})}{dx} = -\sin e^{x^5-3x} e^{x^5-3x}(5x^4 - 3).$$

□

13. P231. #28.14

Proof. **(a).** We know that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Let $h = x - a$. Then $x \to a$ is equivalent to $h \to 0$, and $x = a + h$. So

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

(b). We write

$$\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{h} = \lim_{h\to 0} \frac{(f(a+h) - f(a)) + (f(a) - f(a-h))}{2h}$$
$$= \frac{1}{2}\left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}\right)$$
$$= f'(a).$$

□

14. P231. # 28.16

Proof. "⇒". We define

$$\epsilon(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} - f'(a), & \text{for } x \neq a, \\ 0, & \text{for } x = a. \end{cases}$$

Then

$$f(x) - f(a) = (x - a)(f'(a) - \epsilon(x)).$$

and since f is differentiable at a,

$$\lim_{x\to a} \epsilon(x) = 0.$$

"⇐. " For $x \neq a$,

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a}(f'(a) - \epsilon(x)) = f'(a).$$

So f is differentiable at a and its derivative is $f'(a)$. □

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