HOMEWORK 9

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Abstract. Please send me an email if you find mistakes. Thanks.

1. P289. # 33.1

Proof. The proof is similar to Theorem 33.1. So we skip it. □

2. P289. # 33.2

Proof. We show that $\sup\{cS\} = c \sup S$ for $c > 0$. The proof for infimum is similar. Firstly for $s \in S$,

$$cs \leq c \sup S.$$

So $c \sup S$ is an upper bound. For $\epsilon > 0$, there exists $s_0 \in S$ such that

$$s_0 \geq \sup S - \epsilon/c.$$

So

$$cs_0 \geq c \sup S - \epsilon.$$

This proves that $s \sup S$ is the least one among the upper bounds. Therefore

$$c \sup S = \sup(cS).$$

□

3. P289. # 33.4

Proof.

$$f = \begin{cases} 1, & \text{for } x rational numbers in [0,1], \\ -1, & \text{for } x irrational numbers in [0,1]. \end{cases}$$

One can compute the lower integral $L(f) = -1$ and $U(f) = 1$ as in the book. So f is not integrable. However,

$$|f = 1|$$

is a constant function on $[0, 1]$ and so is integrable. □

4. P289. # 33.5

Proof. Here we use

$$\left|\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)dx\right| \le \int_{-2\pi}^{2\pi} \left|x^2 \sin^8(e^x)\right| dx \le \int_{-2\pi}^{2\pi} x^2 dx = \frac{16\pi^3}{3}.$$

□

5. P289. # 33.7

Proof. **(a).** For any partition $P = \{t_0 = a < t_1 < t_2 < \cdots < t_n = b\}$ and any $\epsilon > 0$, there exist x_k, y_k such that

$$M(f^2, [t_{k-1}, t_k]) - \epsilon < f^2(x_k)$$

and

$$f^2(y_k) \le m(f^2, [t_{k-1}, t_k]) + \epsilon.$$

Then

$$M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \le f^2(x_k) - f^2(y_k) + 2\epsilon = (f(x_k) + f(y_k))(f(x_k) - f(y_k)) + 2\epsilon.$$

Then

$$\begin{aligned}
&\left|M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k])\right| \\
&\le |(f(x_k) + f(y_k)| \, |f(x_k) - f(y_k)| + 2\epsilon \\
&\le 2B|f(x_k) - f(y_k)| + 2\epsilon \le 2B\left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\right) + 2\epsilon.
\end{aligned}$$

Therefore

$$M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \le 2B\left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\right) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \le 2B\left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\right).$$

This implies

$$U(f^2, P) - L(f^2, P) \le 2B\left(U(f, P) - L(f, P)\right).$$

(b). f is integrable on $[a, b]$: for any $\epsilon > 0$, there exists P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon/2B.$$

So

$$U(f^2, P) - L(f^2, P) < \epsilon.$$

This proves that f^2 is integrable on $[a, b]$. □

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