HW 1

1. Let E^0 denote the set of all interior points of a set E. E^0 is called the interior of E.

(a). Prove that E^0 is always open.
(b). Prove that E^0 is open if and only if $E = E^0$.
(c). If $G \subset E$ and if G is open, prove that $G \subset E^0$.
(d). Prove that the complement of E^0 is the closure of the complement of E.
(e). Do E and $\bar{E}$ always have the same interiors?
(f). Do E and E^0 always have the same closures?

2. Let X be an infinite set. For $p, q \in X$, define

$$d(p,q) = \begin{cases} 1, & \text{if } p \neq q, \\ 0, & \text{if } p = q. \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

3. Regard Q, the set of all rational numbers, as a metric space, with $d(p,q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in Q?

4.

(a). If A and B are disjoint closed sets in some metric space X, prove that they are separated.
(b). Prove the same for disjoint open sets.
(c). Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B to be the set of all $q \in X$ for which $d(p,q) > \delta$. Prove that $A \cap B = \emptyset$.

5. A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

6. A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X the following is true: For every $x \in X$ and every open set $G \subset X$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a sub-collection of $\{V_\alpha\}$. Prove that every separable metric space has a countable base.

7. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

8. Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments.

9. Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(a). Prove that ρ is a metric on X.

(b). Show that a subset U of X is open with respect to the metric d if and only if it is open with respect to the metric ρ.

10. Prove that every compact metric space K has a countable base, and that K is therefore separable.